VCU, Department of Computer Science

CMSC 302

11 Advanced Counting Techniques Recurrences

Vojislav Kecman

Ch 7 in 6th edition
Ch 8 in 7th edition

Recurrence Relations
a.k.a. **discrete equations** in system theory

- Definition and Examples
- Solving Recurrences
  - linear homogeneous recurrence
  - linear nonhomogeneous recurrence

Recurrence Relations

- A *recurrence relation* (R.R., or just *recurrence*) for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous elements $a_0, \ldots, a_{n-1}$ of the sequence, for all $n \geq n_0$.

- A particular sequence (described non-recursively i.e., given in a closed form) is said to ***solve*** the given recurrence relation if it is consistent with the definition of the recurrence.
  - A given recurrence relation may have **infinite number** of **solutions**.

Recurrence Relations

•In other words, a recurrence relation is like a recursively defined sequence.

• **Without specifying initial values (initial conditions).** the same recurrence relation can have (and usually has) **infinite number of solutions**.

•If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Recurrence Relation Example

- Consider the recurrence relation
  $a_n = 2a_{n-1} - a_{n-2}$ $(n \geq 2)$.

- Which of the following are solutions?
  1. $a_n = 3n$
  2. $a_n = 2^n$
  3. $a_n = 7$

Recurrence solutions

- $a_n = 2a_{n-1} - a_{n-2}$ **for** $n \geq 2$
- Is $a_n = 3n$ a solution?
  $3n =?$ $2\cdot3(n-1) - 3(n-2) = 6n-6-3n+6 = 3n$ YES

- Is $a_n = 2^n$ a solution?
  $2^n =?$ $2\cdot2^{n-1} - 2^{n-2} = 2\cdot(1/2)\cdot2^n - (1/4)2^n =$
  $(3/4)\cdot2^n$ NO

- Is $a_n = 7$ a solution?
  $7 =?$ $2\cdot7 - 7 = 7$ YES

Example: compound interest

- Suppose you deposit P_0 dollars in a savings account with a fixed interest rate of 5%. How much money do you have after *n* years?
  (assume no withdrawals and no taxes)

- $P_n = P_{n-1} + 5\%$ of $P_{n-1} = P_{n-1} + 0.05\, P_{n-1} = 1.05\, P_{n-1}$

- $P_n = 1.05\,(1.05\, P_{n-2}) = 1.05\,(1.05\,(1.05\, P_{n-3})) = \ldots = 1.05^n\, P_0$

Solving Compound Interest RR

$$
\begin{aligned}
M_n &= M_{n-1} + (P/100)M_{n-1} \\
&= (1 + P/100)\, M_{n-1} \\
&= r\, M_{n-1} \qquad \text{(let } r = 1 + P/100\text{)} \\
&= r\,(r\, M_{n-2}) \\
&= r\cdot r\cdot(r\, M_{n-3}) \qquad \ldots\text{and so on to}\ldots \\
&= r^n\, M_0
\end{aligned}
$$

More Examples

- Growth of a population in which each organism yields 1 new one every period starting 2 periods after its birth.

  $P_n = P_{n-1} + P_{n-2}$ (Fibonacci relation)

Until now we have seen LINEAR RR only! What about these ones?

Nonlinear recurrence relations

$$a_n = a_{n-1}a_{n-2}$$

$$a_n = a_{n-1}^2 + 3a_{n-2}$$

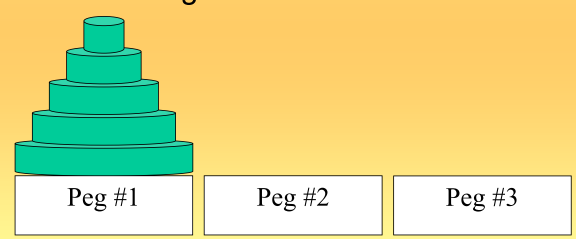
$$a_n = 2a_{n-1} + 3\sin(a_{n-2})$$

More Examples
Tower of Hanoi Example 1

- A nineteen century puzzle created by a French mathematician
- There are **three pegs** and ***n*** disks of different size. The disk are placed in order of size on the first peg, with the largest disk at the bottom. Disks can be moved one at a time to an empty peg or **on top of a larger disk**
- Goal: Move all disks to peg # 3 in a minimal number of moves in order of size!

More Examples
Tower of Hanoi Example 2

- Problem: Get all disks from peg 1 to peg 3.
  - Only move 1 disk at a time.
  - Never set a larger disk on a smaller one.

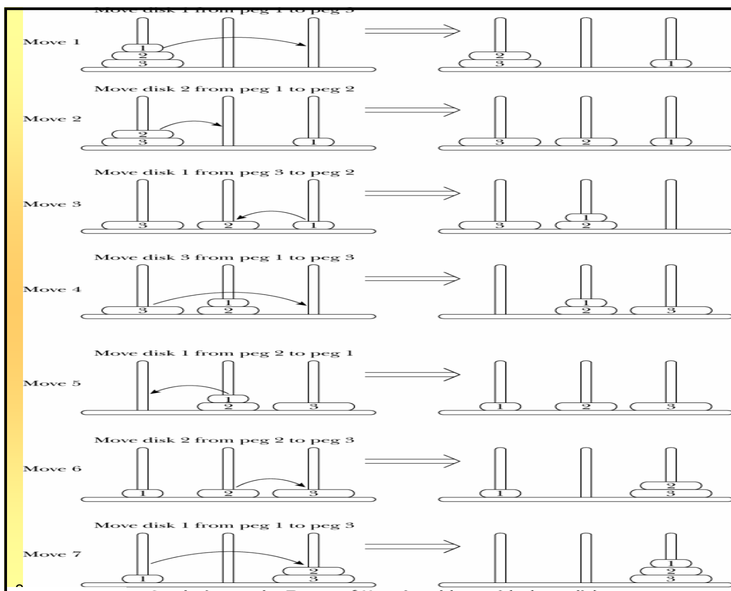


Tower of Hanoi Example - Strategies 1

- First consider the case in which the first peg contains only one disk.
  - The disk can be moved directly from peg 1 to peg 3
- Consider the case in which the first peg contains two disks.
  - First move the first disk from peg 1 to peg 2.
  - Then move the second disk from peg 1 to peg 3.
  - Finally, move the first disk from peg 2 to peg 3.

Tower of Hanoi Example - Strategies 2

- Consider the case in which the first peg contains three disks and then generalize this to the case of 64 disks (in fact, to an arbitrary number of disks).
  - Suppose that peg 1 contains three disks. To move disk number 3 to peg 3, the top two disks must first be moved to peg 2. Disk number 3 can then be moved from peg 1 to peg 3. To move the top two disks from peg 2 to peg 3, use the same strategy as before. This time use peg 1 as the intermediate peg.
  - Figure on next slide shows a solution to the Tower of Hanoi problem with three disks.



A solution to the Tower of Hanoi problem with three disks

Solving the ToH puzzle 1

- We observe that at some point we will need to move the bottom (largest) disk
- In order to do so, all other disks will need to be off the original peg or the peg where the largest disk will go, i.e., on the third peg
- Once this is achieved, we can move the largest disk and we can practically ignore it – then move the remaining disks

Solving the ToH puzzle 2

- Let H_n = # moves for a stack of n disks.
- **Optimal** strategy:
  - Move top $n-1$ disks to spare peg. (H_{n-1} moves)
  - Move bottom disk. (1 move)
  - Move top $n-1$ to bottom disk. (H_{n-1} moves)
- Note: $H_n = 2H_{n-1} + 1$

Solving the ToH puzzle 3

$$
\begin{aligned}
H_n &= 2\,H_{n-1} + 1 \\
&= 2\,(2\,H_{n-2} + 1) + 1 && = 2^2\,H_{n-2} + 2 + 1 \\
&= 2^2(2\,H_{n-3} + 1) + 2 + 1 && = 2^3\,H_{n-3} + 2^2 + 2 + 1 \\
&\ldots \\
&= 2^{n-1}\,H_1 + 2^{n-2} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \ldots + 2 + 1 && \text{(since } H_1 = 1\text{)}
\end{aligned}
$$

$$= \sum_{i=0}^{n-1} 2^i = 2^n - 1$$

This is a closed form expression

Note, this is the equation for an **optimal number of moves!!!**

ToH - Number of moves

- For our proposed solution
  - $H_n = 2H_{n-1} + 1$, $H_1 = 1$
- Is this the only solution?
  - NO. For example, we can make extra moves of the top disks in any peg and back
- Is there another solution with fewer moves?
  - NO because of the reasoning when we first presented the solution

Complexity of the ToH puzzle

- Associated folklore stated that monks were actually working this puzzle in a tower in Hanoi using **64 gold disks**, and the world would end when they solved it
- How much time would that take?
- $H_{64} = 2^{64}-1 \approx 16\cdot 2^{60} = 16\cdot(2^{10})^6 \approx 16\cdot(10^3)^6 = 16\cdot 10^{18}$ moves
- If a move takes a second, about **500 billion years !!!???!!!**

Towers of Hanoi

- Interactive version where you can move the disks around
  - https://romek.info/games/hanoi4e.html
  - http://zylla.wipos.p.lodz.pl/games/hanoi4e.html

If you want to have all disks moved to the Peg # 3 remember the simple rule

- if the number of disks is EVEN the first disk goes to peg # 2,
- if the number of disks is ODD the first disk goes to peg # 3.

Review Questions

- Find the terms a_3 and a_4 of the sequence $\{a_n\}$ where $a_n = a_{n-1}^2 + 2a_{n-2}$, $a_0=1$, $a_1=1$.
- Which of the following sequences are solutions of the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$? (a) $a_n=0$; (b) $a_n=2$; (c) $a_n=4^n$.
- A colony of bacteria triples in size every hour. Find a recurrence relation for its size and the solution of this recurrence relation.

Basics of the polynomials

Every n-order polynomial with **real coefficients** a_i

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0$$

can be expressed as the product of n monomials

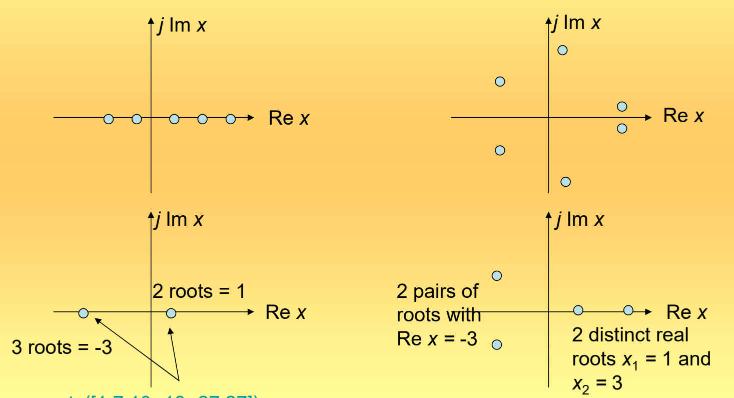
$$(x - x_1)(x - x_2) \ldots (x - x_{n-1})(x - x_n) = 0$$

where x_i are the roots of the polynomial.

If there is a complex root x_i there will always be its conjugate x_j, meaning **complex roots appear only in pairs**.

Basics of the roots of polynomials

Every n-order polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0$ has n roots, which can be real or imaginary, distinct or repeated



roots([1 7 10 -18 -27 27])

Basics of finding the roots of polynomials

1st order polynomial: $ax + b = 0$ $\quad x = -\frac{b}{a}$

2nd order polynomial: $ax^2 + bx + c = 0$ $\quad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

3rd order polynomial: $ax^3 + bx^2 + cx + d = 0$

There are ways – Cardano's method, Vieta's substitution etc, but too complicated expressions!

If in need find by trying (if possible) one solution, then use long division to get a 2nd order polynomial to solve for the last two roots.

Solving Recurrences

- A *linear homogeneous recurrence of degree **k** with constant coefficients* ("*k*-LiHoReCoCo") is a recurrence of the form

  $a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}$,

  where the c_i are all real, and $c_k \neq 0$.

- The solution is **uniquely determined** if ***k*** **initial conditions** $a_0 \ldots a_{k-1}$ are provided.

NOTE VERY CAREFULLY: All what comes is valid for **LINEAR**, **HOMOGENEOUS**, recurrences, with **CONSTANT** coefficients!!!

What is the order *k* of LiHoReCoCo?

- ***k*** is a difference between the biggest and smallest index of LiHoReCoCo
- ***k* is 2** for $a_n = a_{n-1} + 2a_{n-2}$
- ***k* is 7** for $a_{n+2} + a_{n-1} + 2a_{n-5} = 0$
- ***k* is 5** for $a_{n-2} + a_{n-1} + 2a_{n-6} = 0$

For example, a theory of LiHoReCoCo that follows, is not valid for the equations below! It's of no use whatsoever!!!

	WHY?
$a_n = c_1 a_n a_{n-1} + c_2 a_{n-2}$,	**Nonlinearity**
$a_n = n a_{n-1} + \ldots + c_k a_{n-k}$,	**Not constant coefficients**
$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + 3$	**Not homogenous**
$a_n = c_1 a^2_{n-1} + \ldots + c_k a_{n-k}$,	**Nonlinearity**
$a_n = c_1 a_{n-1} + \ldots + (n-k) a_{n-k}$,	**Not constant coefficients**
$a_n = c_1 a_{n-1} + \sin(a_{n-k})$	**Nonlinearity**

Solving LiHoReCoCos

- Basic idea: Look for solutions of the form $a_n = r^n$, where r is a constant.
- Plug r^n into LiHoReCoCos and you get the ***characteristic equation***:
  $r^n = c_1 r^{n-1} + \ldots + c_k r^{n-k}$, if we $/* r^{-n} r^k$
- *we get*
  $r^k - c_1 r^{k-1} - \ldots - c_k = 0$
- The solutions (*characteristic roots*) can yield an explicit formula for the sequence.

Solving 1-LiHoReCoCos

Consider the 1-LiHoReCoCo: $a_n = 2a_{n-1}$, $a_0 = 2$

Start with $a_n = Cr^n$ and plug it in here and

we get $\mathbf{r^n - 2r^{n-1} = r^n - 2r^n/r = 0 \quad /r^n}$

$\mathbf{r - 2 = 0}$ and $\mathbf{r = 2}$

$\mathbf{a_n = C2^n}$ now, to get C we use initial conditions, and write

$a_0 = 2 = C2^0$ → C = 2, and the solution is

$a_n = 2*2^n = 2^{n+1}$. Check it by plugging it back here

Solving 2-LiHoReCoCos

- Consider an arbitrary 2-LiHoReCoCo:
  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$
- It has the characteristic equation (C.E.):
  $r^2 - c_1 r - c_2 = 0$
- **Theorem 1**: If this CE has **2 different roots $r_1 \neq r_2$**, then
  $a_n = Ar_1^n + Br_2^n$ for $n \geq 0$
  for some constants A, B.
- Note that ***A* and *B* are uniquely defined by INITIAL CONDITIONS ONLY**

Example

- Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ given the initial conditions $a_0 = 2$, $a_1 = 7$.
- Solution: First rewrite recurrence as
  - $a_n - a_{n-1} - 2a_{n-2} = 0$ i.e., as $r^n - r^{n-1} - 2 r^{n-2} = 0$
  - Which leads to the characteristic equation:
    $r^2 - r - 2 = 0$
  - Solutions: $r = [-(-1) \pm ((-1)^2 - 4 \cdot 1 \cdot (-2))^{1/2}] / 2 \cdot 1$
    $= (1 \pm 9^{1/2})/2 = (1 \pm 3)/2$, so $\mathbf{r_1 = 2}$ and $\mathbf{r_2 = -1}$.
  - So $a_n = A\, 2^n + B\, (-1)^n$.

What about A and B? **Now initial conditions should jump in!!!**

Example Continued…

- To find A and B, solve the equations for the initial conditions a_0 and a_1:

  $a_0 = 2 = A2^0 + B(-1)^0$

  $a_1 = 7 = A2^1 + B(-1)^1$
- Simplifying, we have the pair of equations:

  $2 = A + B$

  $7 = 2A - B$

  which we can solve easily by addition:

  $9 = 3A$; $A = 3$; $B = -1$.
- Final answer: $a_n = 3 \cdot 2^n - (-1)^n$

Example - Solving Fibonacci

Remind, recipe solution has 3 basic steps:

1) Assume solution of the form $a_n = r^n$
2) Find all possible r's that seem to make this work. Call these r_1 and r_2. Modify assumed solution to ***general solution*** $a_n = Ar_1^n + Br_2^n$ where A,B are constants.
3) Use initial conditions to find A,B and obtain specific solution.

Solving Fibonacci

1) Again, assume exponential solution of the form $a_n = r^n$:

Plug this into $a_n = a_{n-1} + a_{n-2}$:

$$r^n = r^{n-1} + r^{n-2}$$

Notice that all three terms have a common r^{n-2} factor, so divide this out:

$$r^n / r^{n-2} = (r^{n-1} + r^{n-2}) / r^{n-2} \rightarrow r^2 - r - 1 = 0$$

This equation is the ***characteristic equation*** of the Fibonacci recurrence relation.

Solving Fibonacci

2) Find all possible r's that solve characteristic

$$r^2 = r + 1$$

Call these r_1 and r_2. General solution is $a_n = Ar_1^n + Br_2^n$ where A, B are constants.

Quadratic formula gives:

$$r = (1 \pm \sqrt{5})/2$$

So $r_1 = (1+\sqrt{5})/2$, $r_2 = (1-\sqrt{5})/2$

General solution:

$$a_n = A\,[(1+\sqrt{5})/2]^n + B\,[(1-\sqrt{5})/2]^n$$

Solving Fibonacci

3) Use initial conditions $a_0 = 0$, $a_1 = 1$ to find A,B and obtain specific solution.

$0=a_0 = A\,[(1+\sqrt{5})/2]^0 + B\,[(1-\sqrt{5})/2]^0 = A + B$

$1=a_1 = A\,[(1+\sqrt{5})/2]^1 + B\,[(1-\sqrt{5})/2]^1 =$

$A(1+\sqrt{5})/2 + B(1-\sqrt{5})/2 =$

$(A+B)/2 + (A-B)\sqrt{5}/2$

First equation give $B = -A$. Plug into 2nd:

$1 = 0 + 2A\sqrt{5}/2$ so $A = 1/\sqrt{5}$, $B = -1/\sqrt{5}$

Final answer:

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(CHECK IT!)

Solving Fibonacci

3) Use initial conditions $a_1 = 1$, $a_2 = 1$ to find A,B and obtain specific solution.

$1=a_1 = A\,[(1+\sqrt{5})/2]^1 + B\,[(1-\sqrt{5})/2]^1 = Aa_1 + Ba_2$

$1=a_2 = A\,[(1+\sqrt{5})/2]^2 + B\,[(1-\sqrt{5})/2]^2 = Aa_1^2 + Ba_2^2$

$A(1+\sqrt{5})/2 + B(1-\sqrt{5})/2 =$

$(A+B)/2 + (A-B)\sqrt{5}/2$

First equation give $B = -A$. Plug into 2nd:

$1 = 0 + 2A\sqrt{5}/2$ so $A = 1/\sqrt{5}$, $B = -1/\sqrt{5}$

Final answer:

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(CHECK IT!)

***k*-LiHoReCoCos**

- Consider a k-LiHoReCoCo: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$
- It's C.E. is: $r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$
- Theorem 3: If C.E. has ***k* distinct** roots r_i, then the solutions to the recurrence are of the form:

$$a_n = \sum_{i=1}^{k} A_i r_i^n$$

for all $n \geq 0$, where the **A_i are constants to be determined from the initial conditions**.

***k*-LiHoReCoCos**

- Example: Consider a 4-LiHoReCoCo:

$$a_n = -10a_{n-1} - 35a_{n-2} - 50a_{n-3} - 24a_{n-4}$$
$$a_n + 10a_{n-1} + 35a_{n-2} + 50a_{n-3} + 24a_{n-4} = 0$$

- It's C.E. is: $r^4 + 10r^3 + 35r^2 + 50r + 24 = 0$
- The roots of C.E. are r_1=-1, r_2=-2, r_3=-3, and r_4=-4, and the homogeneous solution is

$$a_n = \sum_{i=1}^{4} A_i r_i^n = A_1(-1)^n + A_2(-2)^n + A_3(-3)^n + A_4(-4)^n$$

for all $n \geq 0$, where the A_i are constants depending upon initial conditions.

Check the solutions below

Example 6 (k = 3)

Find the solution of $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions $a_0=2$, $a_1=5$ and $a_2=15$.

Sol :

The roots of $r^3 - 6r^2 + 11r - 6 = 0$ are

$$r_1 = 1, r_2 = 2, \text{ and } r_3 = 3$$

$$\therefore a_n = A_1 \cdot 1^n + A_2 \cdot 2^n + A_3 \cdot 3^n$$

$$\because a_0 = A_1 + A_2 + A_3 = 2$$
$$a_1 = A_1 + 2A_2 + 3A_3 = 5 \quad \Rightarrow \quad A_1 = 1,$$
$$a_2 = A_1 + 4A_2 + 9A_3 = 15 \qquad A_2 = -1, \quad A_3 = 2$$

$$\therefore \ \mathbf{a_n = 1 - 2^n + 2 \cdot 3^n}$$

Homogeneous - Complications

1) ***Repeating* (Degenerate) *roots*** in characteristic equation. Repeating roots imply that they don't learn anything new from second root, so may not have enough information to solve formula with given initial conditions. We'll see how to deal with this on next slide.
2) ***Non-real (complex) numbers roots*** in characteristic equation. If the sequence has periodic behavior, it may get complex roots (for example $a_n = -a_{n-2}$). We won't worry about this case **as long as the complex roots don't repeat** (in principle, same method works as before, **except use complex arithmetic**).

2) above is a complication only for those who working with complex numbers consider a complicated math.

Complication: Repeating Roots

Let $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$ be a recurrence relation with $c_1, c_2, \dots, c_k \in \mathbf{R}$.

Then, the SOLUTION goes as follows

If $r^k - c_1r^{k-1} - c_2r^{k-2} - \dots - c^k = 0$ has

t distinct roots $r_1, r_2, \dots, r_t$

with multiplicities $m_1, m_2, \dots, m_t$ respectively,

where $m_i \geq 1, \forall i$, and $m_1 + m_2 + \dots + m_t = k$,

then
$$\begin{aligned} a_n = \; & (A_{1,0} + A_{1,1} \cdot n + \dots + A_{1,m_1-1} \cdot n^{m_1-1}) r_1^n \\ + \; & (A_{2,0} + A_{2,1} \cdot n + \dots + A_{2,m_2-1} \cdot n^{m_2-1}) \cdot r_2^n \\ + \dots + \; & (A_{t,0} + A_{t,1} \cdot n + \dots + A_{t,m_t-1} \cdot n^{m_t-1}) \cdot r_t^n \end{aligned}$$

where $A_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$ and they are to be found by using the k initial conditions.

Complication: Repeating Roots

EG: Solve $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 1$, $a_1 = 2$

Find characteristic equation by plugging in $a_n = r^n$:

$$r^2 - 2r + 1 = 0$$

Since $r^2 - 2r + 1 = (r-1)^2$ the root $r = 1$ repeats.

If we tried to solve by using general solution

$$a_n = Ar_1^n + Br_2^n = A1^n + B1^n = (A+B)1^n$$

which forces a_n to be a constant function (→←).

SOLUTION: Multiply second solution by n so general solution looks like:

$$a_n = Ar_1^n + Bnr_1^n$$

Complication: Repeating Roots

Solve $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 1$, $a_1 = 2$

General solution: $a_n = A1^n + Bn1^n = A + Bn$

Plug into initial conditions

$1 = a_0 = A + B \cdot 0 \cdot 1^0 = A$

$2 = a_1 = A \cdot 1^1 + B \cdot 1 \cdot 1^1 = A + B$

Plugging first equation $A = 1$ into second:

$2 = 1 + B$ implies $B = 1$.

Final answer: $a_n = 1 + n$

(CHECK IT!)

One more example with repeated roots :

What's the solution of $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution :

The root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$.

Hence $a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n$.

$a_0 = 1 = \alpha_1 \cdot 3^0 + \alpha_2 \cdot 0 \cdot 3^0 \quad \Rightarrow \quad \alpha_1 = 1$

$a_1 = 3\alpha_1 + 3\alpha_2 = 6 \quad \Rightarrow \quad \alpha_2 = 1$

$\Rightarrow a_n = 3^n + n \cdot 3^n$

And again, one more example with repeated roots :

Example Find the solution to the recurrence relation $\boldsymbol{a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Sol :

$r^3 + 3r^2 + 3r + 1 = 0$ has a single root $r_0 = -1$ of multiplicity three.

$\therefore a_n = (A_1 + A_2 n + A_3 n^2)\, r_0^{\,n} = (A_1 + A_2 n + A_3 n^2)(-1)^n$

$\because a_0 = \alpha_1 = 1$

$a_1 = (A_1 + A_2 + A_3) \cdot (-1) = -2$

$a_2 = A_1 + 2A_2 + 4A_3 = -1$

$\therefore A_1 = 1,\ A_2 = 3,\ A_3 = -2$

$\Rightarrow \quad \boldsymbol{a_n = (1 + 3n - 2n^2) \cdot (-1)^n}$

Li<u>NoHo</u>ReCoCos

- Linear ***<u>no</u>nhomogeneous*** RRs with constant coefficients may (unlike Li<u>Ho</u>ReCoCos) contain some terms ***F*(*n*)** that **depend *only* on *n*** (and ***not* on any a_i's**) or ***F*(*n*)** is just **a constant**.

- General form:

$$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + F(n)$$

$$a_n - c_1 a_{n-1} - \ldots - c_k a_{n-k} = F(n)$$

Solutions of LiNoHoReCoCos

- A useful theorem about LiNoHoReCoCos:
  - If $p(n)$ is any ***particular*** solution to the LiNoHoReCoCo
  - Then ***all*** its solutions are of the form:

$$a_n = p(n) + h(n),$$

where $h(n)$ is any solution to the associated **homogeneous** RR

Why is it this way?

- Consider that homogeneous part is given and fixed and only changing part is $F(n)$,
- $a_n = 2a_{n-1} - a_{n-2} + 3 \Rightarrow a_n - 2a_{n-1} + a_{n-2} = 3$
- $a_n = 2a_{n-1} - a_{n-2} + 3^n \Rightarrow a_n - 2a_{n-1} + a_{n-2} = 3^n$
- $a_n = 2a_{n-1} - a_{n-2} + 3n \Rightarrow a_n - 2a_{n-1} + a_{n-2} = 3n$
- $a_n = 2a_{n-1} - a_{n-2} + n^2 \Rightarrow a_n - 2a_{n-1} + a_{n-2} = n^2$
- $a_n = 2a_{n-1} - a_{n-2} + \sin(n) \Rightarrow a_n - 2a_{n-1} + a_{n-2} = \sin(n)$
- Hence, recurrence relations is described with same homogeneous equation and only changing term is right hand side, i.e., the homogeneous solution will be same and each time we will have a particular solution corresponding to a particular $F(n)$

Solution to the Nonhomogeneous Case

Consider the Tower of Hanoi recurrence

$$a_n = 2a_{n-1}+1.$$

Let's solve it methodically. Rewrite:

$$a_n - 2a_{n-1} = 1$$

1) Solve with the RHS set to 0, i.e. solve the homogeneous case.
2) Add a particular solution to get general solution, i.e., use rule:

General Nonhomogeneous Solution = General Homogeneous Solution + Particular Nonhomogeneous Solution

Solution to the Nonhomogeneous Case

$$a_n - 2a_{n-1} = 1$$

1) Solve with the RHS set to 0, i.e. solve first

$$a_n - 2a_{n-1} = 0$$

Characteristic equation: $r - 2 = 0$

so unique root is $r = 2$. General solution to homogeneous equation is

$$a_{nH} = A \cdot 2^n$$

Solution to the Nonhomogeneous Case

2) Add a particular solution to get general solution for $a_n - 2a_{n-1} = 1$. Use rule:

General Nonhomogeneous = General homogeneous + Particular Nonhomogeneous

There are **little tricks** for guessing particular nonhomogeneous solutions. For example, when the RHS is constant, the guess should also be a constant.

So guess a particular solution of the form $b_n = C$.

Plug into the original recursion:

$1 = b_n - 2b_{n-1} = C - 2C = -C$. Therefore $C = -1$.

General solution: $a_n = A \cdot 2^n - 1$.

Solution to the Nonhomogeneous Case

Finally, use initial conditions (I.Cs.) to get closed solution. In the case of the Towers of Hanoi recursion, initial condition is:

$$a_1 = 1$$

Using general solution $a_n = A \cdot 2^n - 1$ we get:

$1 = a_1 = A \cdot 2^1 - 1 = 2A - 1$.

Therefore, $2 = 2A$, so $A = 1$, and

the final answer is

$$\mathbf{a_n = 2^n - 1}$$

Example

- Find all solutions to $a_n = 3a_{n-1} + 2n$. Which solution has $a_1 = 3$?
  - Notice this is a 1-LiNoHoReCoCo. Its associated 1-LiHoReCoCo is $a_n = 3a_{n-1}$, whose solutions are all of the form $a_n = A3^n$.
  - Thus the solutions to the original problem are all of the form $a_n = p(n) + A3^n$.
  - So, all we need to do is find one $p(n)$ that works.

Trial Solutions

- If the extra terms $F(n)$ are a degree-t polynomial in n, you should try a degree-t polynomial as the particular solution $p(n)$.
- This case: $F(n)$ is linear so try $a_n = cn + d$.
- $cn+d = 3(c(n-1)+d) + 2n$ (for all n)
  $(-2c-2)n + (3c-2d) = 2n + 0$ (collect terms)
  and, $c = -1$, $d = -3/2$.
- So, $a_n = -n - 3/2$ is a **particular** solution.
- Check: $a_{n\geq 1} = \{-5/2, -7/2, -9/2, \ldots\}$

Finding a Desired Solution

- From the previous, we know that all general solutions to our example are of the form $a_n = a_{nH} + a_{nP}$:

  $a_n = A3^n - n - 3/2$.

  Solve this for A for the given I.C., $a_1 = 3$:

  $3 = -1 - 3/2 + A3^1$

  $A = 11/6$

  The answer is $a_n = (11/6)3^n - n - 3/2$

PARTICULAR SOLUTIONS – 3 more examples, first 2 are from* :

The general rule that we follow is:

For any amount of terms with the form $\mathbf{k^n}$, we shall let $\mathbf{a_n}$ be $\mathbf{k^n}$ multiplied by a constant. So, if the non-homogeneous part is $\mathbf{a_n = 5^n + 78^n}$, then we let the answer be $\mathbf{a_n = c_1 5^n + c_2 78^n}$, in which $\mathbf{c_1}$ and $\mathbf{c_2}$ are constants to be found. The same goes to the form $\mathbf{nk^n}$, in which you let $\mathbf{a_n = c_1 nk^n}$. However, there is an **exception**, when the root **r** is of the same form as $\mathbf{k^n}$. **We, will not go into these specifics. It's for a more specialized course!**

Example A (terms of the form k^n)

$\mathbf{a_n = 3a_{n-1} + 2^n}$, I.C. is $\mathbf{a_0 = 2}$,

We first proceed to solve the associated homogeneous recurrence relation,

$\mathbf{a_n - 3a_{n-1} = 0}$

The characteristic equation gives us $\mathbf{r = 3}$, and therefore

$\mathbf{a_{nH} = c_1(3^n)}$

Now, after the homogeneous part is solved, we proceed to solve the non-homogeneous part.

Using a smart guess, we let

$\mathbf{a_{nP} = c_2 2^n}$

From here, we then deduce that $\mathbf{a_{n-1P} = c_2 2^{n-1}}$.

Putting the last 2 equations back to the initial recurrence relation which is $\mathbf{a_{nP} = 3a_{n-1P} + 2^n}$, and we have

$\mathbf{c_2 2^n = 3c_2 2^{n-1} + 2^n}$

$\mathbf{(c_2 - 1)2^n = 3c_2 2^{n-1}}$ here we multiply both sides by $\mathbf{2 / 2^n}$ and we obtain

$\mathbf{2(c_2 - 1) = 3c_2}$

$\mathbf{2c_2 - 2 = 3c_2}$

and, we have $\mathbf{c_2 = -2}$,

which then gives us $\mathbf{a_{nP} = -2(2^n) = -2^{n+1}}$.

Now, the GENERAL SOLUTIONS IS THE SUM of homogeneous and non-homogeneous i.e., particular part, and we have

$\mathbf{a_n = c_1(3^n) - 2^{n+1}}$

Only now we apply initial condition(s) I.C, to find $\mathbf{c_1}$. Say, I.C. is $\mathbf{a_0 = 2}$,

$\mathbf{2 = c_1 3^0 - 2^{0+1}}$ or $\mathbf{c_1 = 4}$

So, our FINAL GENERAL SOLUTIONS FOR A GIVEN I.C. is

$\mathbf{a_n = 4(3^n) - 2^{n+1}}$

Example B (nonhomogeneous part are polynomial terms, $an^2 + bn + c$)

$\mathbf{a_n = 3a_{n-1} + n^2 + 5n + 3}$, **I.C. is $\mathbf{a_1 = 0}$.**

(Note that homogeneous solution is same as in Example A, and so is the $\mathbf{a_{nH}}$)

$\mathbf{a_{nH} = c_1(3^n)}$. But, for the non-homogeneous i.e., particular, solution, we assume

$\mathbf{a_{nP} = c_2 n^2 + c_3 n + c_4}$ **(1)**

$\mathbf{a_{n-1P} = c_2(n-1)^2 + c_3(n-1) + c_4}$ **(2)**

You may have got the pattern by now. Note that **if the original equation was**

$\mathbf{a_{nP} = 3a_{n-1} + n^2 + 3}$, or

$\mathbf{a_{nP} = 3a_{n-1} + n^2 + 5n}$, or

$\mathbf{a_{nP} = 3a_{n-1} + n^2}$

we still need to use the above, $\mathbf{a_n = c_2 n^2 + c_3 n + c_4}$.

This is because we need to account for the possibly missing terms which might arise in the particular solution.

Now, same as earlier, we plug in (1) and (2) into original equation

$\mathbf{c_2 n^2 + c_3 n + c_4 - 3(c_2(n-1)^2 + c_3(n-1) + c_4) = n^2 + 5n + 3}$

$\mathbf{-2c_2 n^2 + (6c_2 - 2c_3)n - 3c_2 + 3c_3 - 2c_4 = 1n^2 + 5n + 3}$

and constants are, $\mathbf{c_2 = -1/2,\ c_3 = -4}$ **and** $\mathbf{c_4 = -27/4}$

$\mathbf{a_n = a_{nH} + a_{nP} = c_1(3^n) - 1/2n^2 - 4n - 27/4}$ **. By using I.C.. $\mathbf{0 = 3c_1 - 1/2 - 4 - 27/4}$ and**

$$\boxed{\mathbf{a_n = (11.25/3)(3^n) - 1/2n^2 - 4n - 27/4 = 11.25(3^{n-1}) - 1/2n^2 - 4n - 27/4}}$$

Example C: Repeating Roots + F(n)

Solve $a_n = 2a_{n-1} - a_{n-2} + 2^n$, *I.Cs.* $a_1 = 1$, $a_2 = 2$

Homogeneous solution (see p. 36):

$$a_{nH} = A1^n + Bn1^n = A + Bn$$

However, having F(n) we first need to find particular solution and only then to use I.Cs. to find constants.

Assume $a_{nP} = C2^n$, and if so, $a_{n-1P} = C2^{(n-1)}$ & $a_{n-2P} = C2^{(n-2)}$.

Plug them into original recurrence equation above

$C2^n - 2C2^{(n-1)} + C2^{(n-2)} = 1*2^n$

$(C-2C/2+C/4)*2^n = 1*2^n$ and $C=4$.

$a_n = A + Bn + 4*2^n$. Now only one uses I.Cs. to find A and B

$1 = A+B+4*2$ and $2 = A+2B+4*4$,

$A=0$, $B=-7$

$a_n = -7n + 4*2^n$ => $\boxed{a_n = -7n + 2^{n+2}}$

Checking: $-7n + 2^{n+2} - 2(-7n-7 + 2^{n+1}) - 7n - 14 + 2^n = 2^n$

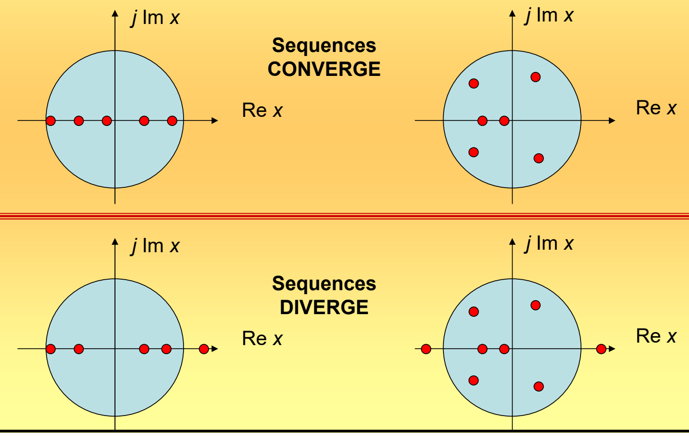
$-7n + 2^{n+2} + 14n + 14 - 2^{n+2} - 7n - 14 + 2^n = 2^n$

$2^n = 2^n$ **Solution is correct**

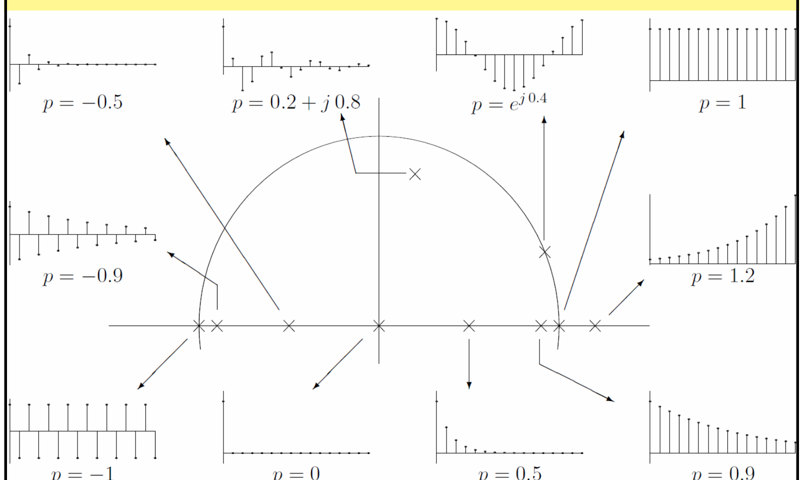
The Nature of Recurrences' Solutions 1

The solutions of LiHOReCoCo with distinc roots is always of this type: $a_n = \sum_{i=1}^{k} A_i r_i^n$

And so, the behavior of a_n depends upon whether **ALL ROOTS** are within the unit circle or not. If former the series **converges**, if later the sequence **diverges**



How each component of the homogeneous solution of an recurrence (sequence) behaves is determined by the recurrence's Ch.Eq. root location as shown below:



Review

- Recurrence relation
- Solving recurrences
  - k-LiHoReCoCos
  - LiNoHoReCoCos